# Modeling Avalanche and Pyroclastic Flows 

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## 1 Introduction

The risk of volcanic eruptions is a problem that public safety authorities throughout the world face several times a year. Volcanic activity can ruin vast areas of productive land, destroy structures, and injure or kill the population of entire cities. The United States Geological Survey reports that globally there are approximately 50 volcanoes that erupt every year. In the 1980s, approximately 30000 people were killed and almost a half million were forced from their homes due to volcanic activity. A 1902 volcanic gravity current from Mt. Pelee, Martinique destroyed the town of St. Pierre and killed all but one of the 29000 inhabitants, the largest number of fatalities from a volcanic eruption in the 20th century. The 1991 eruption of Pinatubo, Philippines, impacted over 1 million people. Hazardous activities consequent to volcanic eruptions range from passive gas emission and slow effusion of lava, to explosions accompanied by the development of a stratospheric plume with associated dense, descending volcanic gravity currents pyroclastic flows! of red-hot ash, rock, and gas that race along the surface away from the volcano. These hot flows can also melt snow on the mountain, creating a muddy mix of ash, water, and rock. Seismic activity at a volcano can trigger the failure of an entire flank of a volcano, generating a giant debris avalanche.

Slower moving mass flows of surficial material take the form of coarse block and ash flows, debris flows, or avalanches. Some of these flows carry with them a significant quantity of water. The 1998 mud flow at Casita Volcano in Nicaragua caused thousands of deaths. Debris flows associated with the 1985 eruption of Nevado del Ruiz, Colombia, resulted in the death of 26000 people. 2 Although scientists had developed a hazard map of the region, the people in the devastated area were unaware of the zones of safety and danger. If they had known, many could have saved themselves. Debris flows originating from severe rainstorms threaten many areas throughout the United States, Mount Rainier being one principal risk site.3,4 In block and ash flows, volcanic avalanches, and debris flows, particles are typically centimeter to meter sized, and the flows, sometimes as fast as hundreds of meters per second, propagate tens of kilometers. As these flows slow, the particle mass sediments out, yielding deposits that can be as much as 100 meters deep and many kilometers in length. For agencies charged with civil protection during volcanic crises, the question they want answered is Should we evacuate a town or village? And if so, when? At present, it is sometimes possible to predict when premonitory activity might lead to a large-scale eruption. It is more difficult to predict when activity might lead to slope failure of some part of the volcano, or the generation of a debris flow. However, one can ask the following question: If a mass flow were to be initiated at a particular location, what areas are most at risk from that flow? This tutorial describes the modeling and computational backbone of the TITAN2D computational environment. TITAN2D solves the "thin layer" equations governing mass flows, analogous to the shallow water equations modeling shallow fluid flows. The numerical basis for the software is a parallel, adaptive grid Godunov solver. Digital elevation maps are imported into the computing environment, the governing equations solved, and output data can produce still images and graphical movies showing flows over a realistic terrain.


Figure 1: A schematic illustration of the forces acting on a mass flowing down an inclined plane.

## 2 Sliding Friction

The simplest consideration is to imagine a solid block of mass $m$ sliding down an inclined plane making an angle $\theta$ with the horizontal. The force down the planar surface is $m g \sin (\theta)$. Friction opposes this motion; the frictional force is proportional to the normal force on the plane, and given by $\mu m g \cos (\theta)$. Newton's Law then reads

$$
\begin{equation*}
m \frac{d u}{d t}=m g \sin (\theta)-\operatorname{sgn}(u) \mu m g \cos (\theta) \tag{1}
\end{equation*}
$$

In writing this equation, we have taken downslope as the positive $x$-direction, and $u$ as the velocity in $x$. The sgn function indicates that friction always opposes motion. Often the friction coefficient is written as $\mu=\tan (\delta)$ where $\delta$ is the dynamic angle of friction. One sees that the mass scales out of the equation, and acceleration is a competition of gravity and friction.

## 3 A Heuristic Derivation

To begin we illustrate the major features of a one-dimensional model of granular flow down an incline. We do so to highlight the important assumptions of such a model. In subsequent sections we provide a mathematically rigorous derivation in higher space dimensions. In the Cartesian plane, consider a thin layer of an incompressible granular material flowing down a flat plane making an angle $\theta$ with the horizontal. Let $[x, b(x)]$ be a point on this plane, and let $z$ be the direction normal to the plane; see Fig. 1 . Assume a constant density of material $\rho$. Consideration of mass conservation for a slice of this layer between the points $x-\Delta x / 2$ and $x+\Delta x / 2$ balances the time derivative of material mass, $\frac{\partial}{\partial t}(\rho h \Delta t)$ and the flux of material $\rho h \mathbf{u}$ across the edges at $x-\Delta x / 2$ and $x+\Delta x / 2$. Taking the limit yields an
equation for the evolution of the height,

$$
\begin{equation*}
\frac{\partial}{\partial t}(\rho h)+\frac{\partial}{\partial x}(\rho h u)=0 \tag{2}
\end{equation*}
$$

This slice is subject to forces in the $x$ and $z$ directions due to gravity, friction, and material deformation, assumed to be characterized by MohrCoulomb plasticity. This MohrCoulomb theory, the generalization of simple sliding friction to a continuum, makes the following assumptions on material behavior.

- Material deforms when the total stress reaches yield, described by the condition $\|\operatorname{dev}(\mathbf{T})\|=$ $\kappa \operatorname{tr}(\mathbf{T})$ where $\mathbf{T}$ is the $2 \times 2$ stress tensor, $\operatorname{dev}$ is the deviator of the tensor $\operatorname{dev}(\mathbf{T})=$ $\mathbf{T}-\frac{1}{2} \operatorname{tr}(\mathbf{T}) \mathbf{I}$, and $\operatorname{tr}(\mathbf{T})$ is the trace of the stress $\operatorname{tr}(\mathbf{T})=\sum_{\mathbf{i}=\mathbf{1}}^{2} \mathbf{T}^{\mathbf{x}_{\mathbf{i}} \mathbf{x}_{\mathbf{i}}}$.
- As material deforms, the stress and strain-rate tensors are aligned, so $\operatorname{dev}(\mathbf{T})=$ $\lambda \operatorname{dev}(\mathbf{V})$ where the strain-rate $V=-\frac{1}{2}\left(\frac{\partial}{\partial x_{i}} u_{j}+\frac{\partial}{\partial x_{j}} u_{i}\right.$, where $\mathbf{u}=\left(u_{1}, u_{2}\right)$ is the two dimensional velocity field.
- Material is rigid if stresses are below yield.

Let us now develop balance laws for $\bar{T}$, the stresses averaged over the layer depth $h$. If motion normal to the basal surface is negligible, the only forces in the $z$ direction are lithostatic and

$$
\frac{\partial}{\partial z} \bar{T}^{z z}=-\rho g \cos (\theta)
$$

Because the stress vanishes on the top surface, we find $\bar{T}^{z z}=\rho g \cos (\theta)(h-z)$, where $h(x)$ is the thickness of the layer above the point $(x, b(x))$. In the $x$ direction, the total time rate of change of net momentum of the slice is given by $\frac{\partial}{\partial t}(\rho h u \Delta x)$, and is balanced by the derivative of the downslope stress $\frac{\partial}{\partial x} \bar{T}^{x x}$, the basal friction $\left.\bar{T}^{x z}\right|_{b}$, and the lateral component of gravity $\rho g h \sin (\theta)$.

In all its details, Mohr - Coulomb plasticity is too complex a theory to apply here. Instead we make a series of simplifying assumptions that allow us to derive a tractable set of equations. First, we assume that material is always deforming, so no region is rigid. Next, at the basal surface we assume perfect alignment of the normal stress and the shear stress, so $\bar{T}^{x z}\left|b=\bar{T}^{z z}\right| b$. We further assume that the stresses remained aligned throughout the layer thickness; any misalignment is likely to be small if the layer is not very thick. Finally, we assume that the $x x$ stress and the $z z$ stress are proportional throughout the layer, $\bar{T}^{x x}=K \bar{T}^{z z} \mid b$; here $K$ is taken to be the earth pressure coefficient, a classical factor that is widely used and whose origins extend back to Rankine [17, 14]. In essence, these assumptions replace the functional relation between stress and strain rate, and the factor $\lambda$ of plasticity, by a state and space-dependent proportionality "constant" together with fixed axes of alignment, a significant simplification of the constitutive theory. Bringing all of these terms together, and using the hydrostatic relation for the normal stress (and thus for all the stresses!), the balance law reads as

$$
\begin{equation*}
\frac{\partial}{\partial t}(\rho h u)+\frac{\partial}{\partial x}\left(\rho h u^{2}+\frac{1}{2} \beta \rho g h^{2}\right)=\rho g h \sin (\theta)-\operatorname{sgn}(u) \cos (\theta) \tan \left(\phi_{b}\right) \rho g h \tag{3}
\end{equation*}
$$

Here the coefficient $\beta=K \cos (\theta)$ and $\phi_{b}$ is the basal frictional angle.
The earth pressure coefficient $K$ may not be constant, but depends on whether the local downslope and cross-slope flows are expanding or contracting (i.e., in the active or passive states, respectively). That is, $K$ may depend on whether $\frac{\partial}{\partial x} u>0$ or $\frac{\partial}{\partial x} u<0$ at a particular location. Subsequent sections provide a specific definition and a more complete discussion of $K$. We note that a fuller accounting for shearing stresses and slope changes would introduce an additional source term proportional to $u^{2}$. See [6] for a derivation that includes this term.

To better appreciate the relative sizes of terms in this model system, the equations should be scaled in both dependent and independent variables. Clearly, overall stresses can be scaled by the lithostatic force $\rho g H \cos (\theta)$, where $H$ is a characteristic thickness of the flowing layer. This scaling removes the density as a parameter, and clears some of the trigonometric functions. But more important is a scaling of the independent variables. Scale $x$ by $L$, a characteristic downslope length, $z$ by $H$, and $t$ by $\sqrt{L / g}$, and make the long wave assumption, namely $\epsilon=H / L \ll 1$. With these scalings, the equations as presented above are modified by the introduction of $\epsilon$ modifying the pressure-like term $\beta \rho g h^{2}$. Similar to the shallow water equations in structure, this system of equations is strictly hyperbolic and genuinely nonlinear away from the "vacuum state" where $h=0$; the characteristic speeds for the system are $u \pm \sqrt{\beta h}$.

One can identify similar terms in Eqns.(11) and (3). The principal difference is the pressure-like term $\frac{1}{2} \beta \rho g h^{2}$, which has no analogue in the sliding block.

Additional terms can be added to the system. The first addition is due to curvature effects that are activated when the basal surface is no longer flat: $\operatorname{sgn}(u) \tan \left(\phi_{b}\right) \rho g h \frac{u^{2}}{R g_{z}}$ where $R$ is the radius of curvature and $u^{2} / R$ is the centripetal acceleration. The second term is a Voellmy effect, a quadratic drag effect: $\rho d|u| u$, where $d$ is the drag coefficient.

## 4 A More Rigorous Derivation

In this section we consider flows in three space dimensions. We give a somewhat more complete derivation of the thin layer equations. The main mathematical tools called upon in this section are from vector calculus, and the one "trick" we use is a depth averaging process that employs Leibniz rule for interchanging differentiation and integration.

### 4.1 Full Equations in 3D

In three space dimensions, consider a thin layer of granular material with constant specific density $\rho$ flowing over a smooth basal surface. Neglect any erosion of the base. At any location, consider a Cartesian coordinate system $O x y z$, with origin $O$ defined so the plane $O x y$ is tangent to the basal surface and $O z$ the normal direction. In considering flow over variable terrain, there should be no preferential direction in the $x y$-plane, but we do assume a globally consistent orientation of this plane. Write $\mathbf{v}$ for the velocities of the solid constitutient When writing equations in component form, we use superscripts to denote the component.

Mass conservation may be written as

$$
\begin{equation*}
\frac{\partial}{\partial t} \rho+\nabla \cdot(\rho \mathbf{v})=0 \tag{4}
\end{equation*}
$$

The momentum equations take the form:

$$
\begin{equation*}
\rho\left(\frac{\partial}{\partial t} \mathbf{v}+(\mathbf{v} \cdot \nabla) \mathbf{v}\right)=-\nabla \cdot \mathbf{T}+\rho \varphi \mathbf{g} \tag{5}
\end{equation*}
$$

In writing these equations, no a priori assumptions are made about the constitutive behavior of the material, and specification of the stresses is still required. Here we will assume the material behaves as a Coulomb granular material.

Next scale the equations. The characteristic length in the $z$-direction is $H$, and in $x, y$ it is $L$. The timescale is taken to be $t^{2}=L / g$. Typical stresses are on the order of the weight of the material, so scale the solid stresses by $\rho g H$. After clearing the coefficients, the continuity equations is unchanged. Several terms in the momentum equations are multiplied by $\epsilon=H / L$ which is small. Specifically, the momentum balance in the $x, y$ and $z$ directions becomes

$$
\begin{align*}
\frac{\partial}{\partial t} v^{x}+v^{x} \frac{\partial}{\partial x} v^{x}+v^{y} \frac{\partial}{\partial y} v^{x}+v^{z} \frac{\partial}{\partial z} v^{x} & =-\left(\epsilon \frac{\partial}{\partial x} T^{x x}+\epsilon \frac{\partial}{\partial y} T^{x y}+\frac{\partial}{\partial z} T^{x z}\right)+g^{x}  \tag{6}\\
\frac{\partial}{\partial t} v^{y}+v^{x} \frac{\partial}{\partial x} v^{y}+v^{2} \frac{\partial}{\partial y} v^{2}+v^{z} \frac{\partial}{\partial z} v^{y} & =-\left(\epsilon \frac{\partial}{\partial x} T^{x y}+\epsilon \frac{\partial}{\partial y} T^{y y}+\frac{\partial}{\partial z} T^{y z}\right)+g^{y} \\
\epsilon\left(\frac{\partial}{\partial t} v^{z}+v^{x} \frac{\partial}{\partial x} v^{z}+v^{y} \frac{\partial}{\partial y} v^{z}+v^{z} \frac{\partial}{\partial z} v^{z}\right) & =-\left(\epsilon \frac{\partial}{\partial x} T^{x z}+\epsilon \frac{\partial}{\partial y} T^{y z}+\frac{\partial}{\partial z} T^{z z}\right)+g^{z} .
\end{align*}
$$

Note that components of gravity have been scaled by the magnitude $g$, so $\left(g^{x}, g^{y}, g^{z}\right)$ is a unit vector. Usually one would drop all terms of order $\epsilon$, however, as discussed in [15] there is a need to retain the diagonal stress contributions. Because no preferential downslope direction is prescribed, and the flow direction may change during a flow, we retain all the stress terms in both the $x$ - and $y$-directions, dropping only $O(\epsilon)$ terms in the $z$-direction; see also [6].

### 4.2 Mass Balance

The mass balance equation tells us the velocity $\mathbf{v}$ is divergence free. Let us illustrate the depth averaging process with this equation. Assume the flowing material occupies the region between $z=b(x, y)$ and $z=h(x, y)$. Then we integrate:

$$
\begin{equation*}
\int_{b}^{h} \nabla \cdot \mathbf{v} d z=0 \tag{7}
\end{equation*}
$$

We now discuss some mathematical issues that arise in the depth averaging process. Leibniz rule states that, for any continuously differentiable function of two variables (here we use the dummy variables $s, w$ )

$$
\frac{\partial}{\partial w} \int_{b(w)}^{h(w)} F(s, w) d s=\int_{b(w)}^{h(w)} \frac{\partial}{\partial w} F(s, w) d s+F(b(w), w) \frac{\partial b}{\partial w}-F(a(w), w) \frac{\partial a}{\partial w}
$$

The upper free surface $F_{h}(\mathbf{x}, t)=0$ is assumed to be a material surface - that is, the upper edge of the material defines the surface $z=h(x, y, t)$.

$$
\begin{equation*}
\frac{\partial}{\partial t} h+v^{x} \frac{\partial}{\partial x} h+v^{y} \frac{\partial}{\partial y} h-v^{z}=0 \tag{8}
\end{equation*}
$$

Likewise, at the fixed basal surface $F_{b}(\mathbf{x})=0$ flow is tangent to the fixed bed so

$$
\begin{equation*}
v^{x} \frac{\partial}{\partial x} b+v^{y} \frac{\partial}{\partial y} b-v^{z}=0 \tag{9}
\end{equation*}
$$

In arriving at these equations, we have ignored deposition and erosion.
Now using Leibniz and the free surface condition, and after some algebraic manipulation, we find an equation for the total mass

$$
\begin{equation*}
\frac{\partial}{\partial t} \hat{h}+\frac{\partial}{\partial x} \hat{h} \overline{v^{x}}++\frac{\partial}{\partial y} \hat{h} \overline{v^{y}}=0 \tag{10}
\end{equation*}
$$

In writing this equation, the depth averaged velocities $\hat{h} \overline{v^{x}}=\int_{b}^{h} v^{x} d z$, with a similar expression for $\overline{v^{y}}$, and where $\hat{h}=h-b$.

### 4.3 Momentum Balance

Depth averaging the momentum equations is more complicated, owing principally to the vector and tensor quantities involved. The complete derivation is presented in Appendix 1. Here we content ourselves with reporting the results:
$z$-Momentum Observe that, upon setting $\epsilon$ to zero in the $z$-momentum equation, we find

$$
\frac{\partial}{\partial z} T^{z z}=g^{z}
$$

Integrating and using the boundary condition that the upper surface is stress free, we find

$$
T^{z z}=[h-z] g^{z} .
$$

That is, the normal solid stress in the $z$-direction at any height is equal to the weight of the material overburden.

Observe that, owing to scaling, the $z$-velocities have been dropped from the $z$-momentum equations. Of course neglecting motion in the $z$-direction is a central component of the thin layer theory.

We now make use of the alignment and proportionality conditions, as in the 1-dimensional case. This means that the $x x, y y, x z$ and $y z$-stresses are all proportional to $T^{z z}$. Depth averaging we find

$$
\begin{align*}
\frac{\partial}{\partial t} & \left(\hat{h} \overline{v^{x}}\right)+\frac{\partial}{\partial x}\left(\hat{h} \overline{v^{x} v^{x}}\right)+\frac{\partial}{\partial y}\left(\hat{h} \overline{v^{x} v^{y}}\right)  \tag{11}\\
=\quad & -\frac{\epsilon}{2} \frac{\partial}{\partial x}\left(\alpha_{x x} \hat{h}^{2}\left(-g^{z}\right)\right)-\frac{\epsilon}{2} \frac{\partial}{\partial y}\left(\alpha_{x y} \hat{h}^{2}\left(-g^{z}\right)\right) \\
& \left.-\epsilon \alpha_{x x} \frac{\partial}{\partial x} b-\epsilon \alpha_{x y} \frac{\partial}{\partial y} b+\alpha_{x z} \hat{h} \overline{( }-g^{z}\right)+\hat{h} g^{x} .
\end{align*}
$$

The terms containing $-g^{z}$ arise from the stress proportionality. The $y$-solid momentum equation can be written

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(\hat{h} \overline{v^{y}}\right)+\frac{\partial}{\partial x}\left(\hat{h} \overline{v^{x} v^{y}}\right)+\frac{\partial}{\partial y}\left(\hat{h} \overline{v^{y} v^{y}}\right)  \tag{12}\\
& =\quad-\frac{\epsilon}{2} \frac{\partial}{\partial x}\left(\alpha_{x y} \hat{h}^{2}\left(-g^{z}\right)\right)-\frac{\epsilon}{2} \frac{\partial}{\partial y}\left(\alpha_{y y} \hat{h}^{2}\left(-g^{z}\right)\right) \\
& \\
& \\
& \left.\quad-\epsilon \alpha_{x y} \frac{\partial}{\partial x} b-\epsilon \alpha_{y y} \frac{\partial}{\partial y} b+\alpha_{y z} \hat{h} \overline{\overline{( }}-g^{z}\right)+\hat{h} g^{y} .
\end{align*}
$$

We have employed the $\alpha$-notation for convenience $T^{* z}=-\frac{v^{*}}{\|\mathbf{v}\|} \tan \left(\phi_{b}\right) T^{z z} \equiv \alpha_{* z} T^{s z z}$.
These equations are hyperbolic, with two nonlinear wave-speeds and one linearly degenerate one; the system loses strict hyperbolicity at the 'vacuum' state where $h=0$.

Again Voellmy and curvature terms can be added to the system. In the TITAN2D system, we include the curvature effect. For completeness, the equations in TITAN2D are

$$
\begin{equation*}
\partial_{t} \mathbf{U}+\partial_{x} \mathbf{F}+\partial_{y} \mathbf{G}=\mathbf{S} \tag{13}
\end{equation*}
$$

where $\mathbf{U}=\left\{h, h v^{x}, h v^{y}\right\}^{T}, \mathbf{F}=\left\{h v^{x}, h v^{x 2}+\frac{\eta}{2} g_{z} h^{2}, h v^{x} v^{y}\right\}^{T}, \mathbf{G}=\left\{h v^{y}, h v^{x} v^{y}, h v^{y 2}+\right.$ $\left.\frac{\eta}{2} g_{z} h^{2}\right\}^{T}, \mathbf{S}=\left\{0, h g_{x}-\left(\frac{v^{x}}{\sqrt{v^{x}+v^{y^{2}}}}\right) h \tan \left(\phi_{b}\right)\left[g_{z}+v^{x 2} \frac{d b}{d x}\right]-\operatorname{sgn}\left(\partial_{y} v^{x}\right) \partial_{y}\left(\frac{\eta}{2} \sin \left(\phi_{i n t}\right) h^{2} g_{z}\right), h g_{y}\right.$ $\left.-\left(\frac{v^{y}}{\sqrt{v^{x 2}+v^{y 2}}}\right) h \tan \left(\phi_{b}\right)\left[g_{z}+v^{2} \frac{d b}{d y}\right]-\operatorname{sgn}\left(\partial_{x} v^{y}\right) \partial_{x}\left(\frac{\eta}{2} \sin \left(\phi_{i n t}\right) h^{2} g_{z}\right)\right\} . \phi_{\text {int }}$ is the internal friction angle, $\eta=\epsilon K$, and $K\left(\phi_{\text {int }}, \phi_{b}\right)$ is the so-called earth pressure coefficient, a function of the friction angles, and describes the ratio of shear stresses to normal stress within the flowing mass. This formulation is a variant of the original Savage-Hutter equations [15] that incorporate certain refinements due to Iverson [6]. The earth pressure coefficient $K$ is in the active or passive state, depending on whether the downslope and cross-slope flows are expanding or contracting. We modify Iverson's assumption [6] (see also [15, 17]) and define two values for $K$

$$
K=2 \frac{1 \pm\left[1-\cos ^{2}\left(\phi_{\text {int }}\right)\left(1+\tan ^{2}\left(\phi_{b}\right)\right)\right]^{\frac{1}{2}}}{\cos ^{2}\left(\phi_{i n t}\right)}-1
$$

where the + (passive state) applies when flow is converging, that is, if $\partial_{x} v^{x}+\partial_{y} v^{y}<0$, and the - (active state) applies if $\partial_{x} v^{x}+\partial_{y} v^{y}>0$.

We emphasize that, in writing these thin layer equations, we have made the long wave assumption. Thus only $O(L)$ variations in $x$ and $y$ are represented in the system. Higher frequency variations are assumed to be negligible and are not captured by the dynamics.

## 5 Computing Flow Along a Two-Dimensional Surface

TITAN2D is a parallel, adaptive grid, shock capturing method to solve the governing mass flow equations. We begin with a discussion of the integration of digital elevation model data - a description of the local topography into our solver. The synthesis of these computational techniques makes possible the solution of mass flows over a realistic terrain on desktop hardware.

### 5.1 Digital elevation data

A principal feature of TITAN2D is the incorporation of topographical data into our simulation and grid structure. We have written a preprocessing routine in which digital elevation data are imported. These data define a two-dimensional spatial box in which the simulation will occur. Association with terrain is based on the UTM (Universal Transverse Mercator coordinate system) for representing points on the geoid. The raw data provide elevations at specified locations. By using these data, and interpolating between data points where necessary, a rectangular, Cartesian mesh is created. This mesh is then indexed in a manner consistent with our computational solver. The elevations provided on this mesh are then used to create surface normals and tangents, ingredients in the governing PDEs. Finally, the grid data are written out for use, together with simulation ouput, in post-computation visualization. The digital elevation data may be obtained from a number of geographic information system (GIS) sources. We have implemented a version that imports GRASS data (Geographic Resource Analysis Support System), which is then georectified and coded into a grid, in a manner similar to the GRID module of ARC/INFO.

Depending on the specific site under consideration, a typical GIS coarse grid provides blocks of about $100 m \times 100 \mathrm{~m}$ in size with a $\pm 30 \mathrm{~m}$ vertical accuracy, while a fine grid can have blocks of $5 m \times 5 m$ in size with a $\pm 1 m$ vertical accuracy.

### 5.2 Numerical Solver

To solve the governing system of equations we use a parallel, adaptive mesh, Godunov solver. The basic ingredient in the method is an approximate Riemann solver. TITAN2D uses a solver originally due to Davis [1] which in turn is based on the ideas of Harten, Lax and vanLeer [2]. In brief, the dependent variables are considered as cell averages, and their values are advanced by a predictorcorrector method; the source terms are included in these updates, and no splitting - neither for the source nor the multiple dimensions - is necessary. Slope limiting is used to prevent unphysical oscillations. The Davis approximate Riemann solver is a centered scheme, akin to an approach introduced by Rusanov. Appendix 2 contains details of the numerical method.

Adaptive gridding coupled with parallel computing enables the use of very fine grids and provides for simulations of very high fidelity. We have implemented a simple adaptive grid structure that refines cells at a given time step based on indicators derived from the computed solution at the previous time step. Perhaps the simplest indicator to consider is "Is $h=0$ ? That is, all cells containing material are refined. In experiments with this simple indicator, we may also refine all cells immediately adjacent to cells containing material. In this way, a one-cell thick band of refined but empty cells surrounds the refined cells containing material. This additional band of refined cells ensures that no spurious waves are generated as material passes across different grids. Alternatively, enforcing precise conservation at edges where refined and unrefined grids meet also provides a highly accurate computation. We have also experimented with a simple scaled $L^{2}$ norm of the flux around the boundary of each cell as the refinement indicator. In a similar way, if indicators suggest that a coarser
discretization will not adversely affect the solution quality, we remove refined cells. Thus, we are able to maintain the solution quality and track special features, while not making the computation prohibitively expensive. We also monitor the change of the pile height on every time step. When this change is below a user defined threshold, we unrefine the grid locally. Both the refinement and unrefinement indicators are heuristic, and based on experience and physical intuition. We must also define the frequency of grid adaption. Experience leads us to examine the indicators every two time steps to decide whether or not to refine or unrefine. Again, this adaption frequency is heuristic.

Parallelization of adaptively refined grids has been addressed by several researchers over the last several years. To take full advantage of multiprocessor computing, the critical issues are the number and location of cells created and deleted during grid refinement, load balance, and the efficiency of storage and data access. At those cells at the boundary of a processor partition we must add data storage for a layer of "ghost cells" on each subdomain. The data of a cell adjacent to the partition and belonging to one processor are replicated on the other processor. This replication enables us to perform the computations necessary for the Godunov scheme for every cell, without explicit communication. However, the use of ghost cells requires us to synchronize the data among the processors at the end of each time step.

Although we use only regular Cartesian grids in the simulations, our data management scheme is imported from finite element computations, where unstructured grids often arise. Thus our design allows for general grids, and our data structure are not simple arrays. Because all data access operations involve some kind of search, this procedure must be fast and efficient. Our data storage model is a distributed hash table. Data are indexed by an ordering along a space filling curve (SFC), and are then stored in the table. Each cell is mapped to a unique key using its location along the SFC passing through its centroid. This key also provides a unique identifier that is easily generated - typically it is generated directly from geometric coordinates by using bit manipulations. To obtain a decomposition of the problem, we introduce a partitioning of this key space. This induces a distribution of the data sets, i.e., a decomposition of the problem for distributed memory computing. When the grid changes and new cells are introduced, a redistribution of cells among the processors to maintain load balance is achieved by adjusting this key space partition. By adjusting the key space partition after grid refinement/unrefinement stages, good load balance is maintained.

## 6 Some Practical Issues in Computing Thin Layer Flows

The derivation just presented clearly shows the significance of the scale parameter $\epsilon=H / L$ in balancing the magnitude of terms in the governing equations. For computations two difficulties arise. Firstly, $L$ is often an input into the computation and not known a priori. Second, for many flows of interest the actual size of $L$ may not be very large! The first difficulty is surmounted by using a sufficiently high scaling parameter. The second is harder to overcome. What constitutes a sufficiently long flow in order for these equations to provide good approximation is a matter of some judgement. As a guide, flows in which the maximum depth is no more than one-tenth of the typical length are good candidates.

Another matter than must be given careful thought is the estimate of friction inputs. Sometimes the basal and internal friction angles are extrapolated from laboratory tests of material, which may differ significantly from field flow conditions. Other times the angles are fitted by calibration to observed flows. Furthermore, real flows often involve a mixture of solid materials and fluids, a rheology that is usually difficult to characterize but certainly requires more complex multi-phase models (or an unrealistic friction coefficient) to capture the effect of fluidizations. The calibration approach can be profitably used in many circumstances. However it is known that there is a volume dependence of the runout of a flow - larger flows travell further. Thus calibration requires comparison against flows of similar size.

Finally, many difficulties arise from the DEM and the quality of that map. Many DEMs that are readily available have regions where elevations are poorly defined or even undefined! Additionally, terrain often has rapid changes (cliff walls and canyons) which can cause computations of gradients and curvatures to become unstable. Since the equations here involve terms with curvatures - evaluated by post-processing the elevations - these calculations require a filtering of the terrain data to avoid changes in gradient larger than about $70^{\circ}$.

## 7 Appendix 1

Here is the full derivation of the depth-averaged momentum equations. We assume the $z$-equation as 4.3

The scaled equations are 6

$$
\begin{align*}
\frac{\partial}{\partial t} v^{x}+v^{x} \frac{\partial}{\partial x} v^{x}+v^{y} \frac{\partial}{\partial y} v^{x}+v^{z} \frac{\partial}{\partial z} v^{x} & =-\left(\epsilon \frac{\partial}{\partial x} T^{x x}+\epsilon \frac{\partial}{\partial y} T^{x y}+\frac{\partial}{\partial z} T^{x z}\right)+g^{x}  \tag{14}\\
\frac{\partial}{\partial t} v^{y}+v^{x} \frac{\partial}{\partial x} v^{y}+v^{2} \frac{\partial}{\partial y} v^{2}+v^{z} \frac{\partial}{\partial z} v^{y} & =-\left(\epsilon \frac{\partial}{\partial x} T^{x y}+\epsilon \frac{\partial}{\partial y} T^{y y}+\frac{\partial}{\partial z} T^{y z}\right)+g^{y} \\
\epsilon\left(\frac{\partial}{\partial t} v^{z}+v^{x} \frac{\partial}{\partial x} v^{z}+v^{y} \frac{\partial}{\partial y} v^{z}+v^{z} \frac{\partial}{\partial z} v^{z}\right) & =-\left(\epsilon \frac{\partial}{\partial x} T^{x z}+\epsilon \frac{\partial}{\partial y} T^{y z}+\frac{\partial}{\partial z} T^{z z}\right)+g^{z} .
\end{align*}
$$

Solving the $z$-equation yields $T^{z z}(x, y, z)=-g^{z}[h-z]$.
The left-hand side of the $x$-momentum equation is

$$
L H S=\frac{\partial}{\partial t} v^{x}+\frac{\partial}{\partial x} v^{x}+\frac{\partial}{\partial y} v^{x} v^{y}+\frac{\partial}{\partial z} v^{x} v^{z} .
$$

Depth average and use boundary conditions to find

$$
\begin{equation*}
\int_{b}^{h} L H S d z=\frac{\partial}{\partial t} \int_{b}^{h} v^{x} d z+\frac{\partial}{\partial x} \int_{b}^{h} v^{x 2} d z+\frac{\partial}{\partial y} \int_{b}^{h} v^{x} v^{y} d z \tag{15}
\end{equation*}
$$

We use approximations such as $\int_{b}^{h} v^{x 2} d z=\hat{h} \bar{v}^{x} v^{x}=\hat{h} \bar{v}^{x} \bar{v}^{x}$. We make an error in this approximation; this error is small for certain flow profiles, but non-negligible for others.

Now depth average the right hand side:

$$
\begin{equation*}
\int_{b}^{h} R H S d z=\underbrace{-\int_{b}^{h}\left(\epsilon \frac{\partial}{\partial x} T^{x x}+\epsilon \frac{\partial}{\partial y} T^{x y}+\frac{\partial}{\partial z} T^{x z}\right) d z}_{(i)}+\int_{b}^{h} \varphi g^{x} d z \tag{16}
\end{equation*}
$$

We assume the earth pressure relation for the solid phase is employed. To this end, basal shear stresses are assumed to be proportional to the normal stress:

$$
T^{* z}=-\frac{v^{*}}{\|\mathbf{v}\|} \tan \left(\phi_{b}\right) T^{z z} \equiv \alpha_{* z} T^{z z}
$$

where $*$ can be either $x$ or $y$, and the velocity ratio determines the force opposing the motion in the $*$-direction, to the extend that force is mobilized [3, 9] in that direction. Thus sliding velocity and basal traction are colinear. The $\alpha$ notation will provide a convenient shorthand. Likewise the diagonal stresses are taken to be proportional to the normal solid stress

$$
T^{* *}=k_{a p} T^{z z} \equiv \alpha_{* *} T^{z z},
$$

where the same index $x$ or $y$ is used in both $*$ 's. Finally, following [6], $x y$ shear stresses are determined by a Coulomb relation

$$
T^{x y}=-\operatorname{sgn}\left(\frac{\partial}{\partial y} v^{x}\right) \sin \left(\phi_{i n t}\right) k_{a p} T^{z z} \equiv \alpha_{x y} T^{z z}
$$

where the sgn function ensures that friction opposes straining in the $(x, y)$-plane.
Now

$$
\begin{align*}
(i)= & -\epsilon \int_{b}^{h} \frac{\partial}{\partial x} \alpha_{x x} T^{z z} d z-\epsilon \int_{b}^{h} \frac{\partial}{\partial y} \alpha_{x y} T^{z z} d z-\int_{b}^{h} \frac{\partial}{\partial z} \alpha_{x z} T^{z z} d z  \tag{17}\\
= & -\epsilon\left[\frac{\partial}{\partial x} \int_{b}^{h} \alpha_{x x} T^{z z} d z-\left.\alpha_{x x} T^{z z}\right|_{z=h} \frac{\partial}{\partial x} h+\left.\alpha_{x x} T^{z z}\right|_{z=b} \frac{\partial}{\partial x} b\right] \\
& -\epsilon\left[\frac{\partial}{\partial y} \int_{b}^{h} \alpha_{x y} T^{z z} d z-\left.\alpha_{x y} T^{z z}\right|_{z=h} \frac{\partial}{\partial y} h+\left.\alpha_{x y} T^{z z}\right|_{z=b} \frac{\partial}{\partial y} b\right] \\
& -\alpha_{x z}\left[\left.T^{z z}\right|_{z=h}-\left.T^{z z}\right|_{z=b}\right] .
\end{align*}
$$

Because the upper free surface is stress free, all terms $\left.T^{z z}\right|_{z=h}$ vanish. Combining all terms yields an $x$-momentum equation:

$$
\begin{align*}
\frac{\partial}{\partial t} & \left(\hat{h} \overline{v^{x}}\right)+\frac{\partial}{\partial x}\left(\hat{h} \bar{v}^{x} v^{x}\right)+\frac{\partial}{\partial y}\left(\hat{h} \overline{v^{x} v^{y}}\right)  \tag{18}\\
=\quad & -\frac{\epsilon}{2} \frac{\partial}{\partial x}\left(\alpha_{x x} \hat{h}^{2}\left(-g^{z}\right)\right)-\frac{\epsilon}{2} \frac{\partial}{\partial y}\left(\alpha_{x y} \hat{h}^{2}\left(-g^{z}\right)\right) \\
& \left.-\epsilon \alpha_{x x} \frac{\partial}{\partial x} b-\epsilon \alpha_{x y} \frac{\partial}{\partial y} b+\alpha_{x z} \hat{h} \overline{( }-g^{z}\right)+\hat{h} g^{x} .
\end{align*}
$$

The $y$-momentum equation is derived similarly.

## 8 Appendix 2

This appendix contains details regarding the finite volume Godunov solver used in TITAN2D.
To solve the mass and momentum balance laws, we use a parallel, adaptive mesh, Godunov solver. The basic ingredient in the method is an approximate Riemann solver. We have coded a solver originally due to Davis [1] based on the ideas of Harten, Lax and vanLeer [2]. In brief, the dependent variables are considered as cell averages, and their values are advanced by a predictorcorrector method; the source terms are included in these updates, and no splittingneither for the source nor the multiple dimensionsis necessary. Slope limiting is used to prevent unphysical oscillations. The Davis approximate Riemann solver is a centered scheme, akin to an approach introduced by Rusanov. Consider, then, a hyperbolic system written as

$$
\begin{equation*}
\frac{\partial}{\partial t} U+\frac{\partial}{\partial x} f(U)+\frac{\partial}{\partial y} G(U)=S(U) \tag{19}
\end{equation*}
$$

We have occasion to use the non-conservative form

$$
\begin{equation*}
\frac{\partial}{\partial t} U+A \frac{\partial}{\partial x} U+B \frac{\partial}{\partial y} U=S(U) \tag{20}
\end{equation*}
$$

where $A, B$ are the Jacobians of $f, g$, respectively. Discretize the variable $U=U_{i j}$ where $U_{i j}$ is considered to be the cell average of $U(x, y)$ in the cell $\left[\left(i-\frac{1}{2}\right) \Delta x,\left(i+\frac{1}{2}\right) \Delta x\right] \times[(j-$ $\left.\left.\frac{1}{2}\right) \Delta y,\left(j+\frac{1}{2}\right) \Delta y\right]$ where for simplicity we assume a uniiform grid. Write $U_{i j}^{n}$ for $U_{i j}$ at time $n \Delta t$. The the midtime predictor is

$$
\begin{equation*}
U_{i j}^{n+\frac{1}{2}}=U_{i j}^{n}+\frac{1}{2} \Delta t A_{i j}^{n} \Delta_{x} U_{i j}^{n}+\frac{1}{2} \Delta t B_{i j}^{n} \Delta_{y} U_{i j}^{n}+\frac{1}{2} \Delta t S_{i j}^{n} \tag{21}
\end{equation*}
$$

In this formula, $\Delta_{x} U$ and $\Delta_{y} U$ are the limited slopes for $U$ in the $x$ - and $y$-directions, respectively (see below). Given the mid-time cell-center value $U_{i j}^{n+\frac{1}{2}}$, extrapolate to the cell edge using the limited slope. For example, $U_{i+\frac{1}{2} j}^{n+\frac{1}{2}}=U_{i j}^{n+\frac{1}{2}}+\frac{1}{2} \Delta_{x} U_{i j}^{n}$ if we extrapolate from cell $i j$ (referred to as the left state), or $U_{i+\frac{1}{2} j}^{n+\frac{1}{2}}=U_{i+1 j}^{n+\frac{1}{2}}-\frac{1}{2} \Delta_{x} U_{i+1 j}^{n}$ if we extrapolate from cell $i+1, j$ (the right state). So now there are two values at the cell edge $i+\frac{1}{2} j$. To resolve this multivaluedness, an approximate Riemann solver generates a numerical flux $F\left(U^{l}, U^{r}\right)$ depending on these left and right states and the physical flux f . The interested reader may consult [1, 2], among other sources, for a guide to the vast literature of Riemann solvers. We follow the early work of Davis who generates a flux by the examination of the fastest and slowest wave speeds propagating from the local states:

$$
\begin{equation*}
F\left(U^{l}, U^{r}\right)=\frac{1}{2}\left[f\left(U^{l}\right)+f\left(U^{r}\right)\right]-\frac{\alpha}{2}\left(U^{r}-U^{l}\right) \tag{22}
\end{equation*}
$$

In this formula, $\alpha$ is an upper bound on the magnitude of the characteristic speeds of all waves (normal to the cell edge under consideration) evaluated at both the left and right states.

Finally, a conservative updated of $U$ is computed as

$$
\begin{align*}
U_{i j}^{n+1}= & U_{i j}^{n}-\frac{\Delta t}{\Delta x}\left[F\left(U_{i+\frac{1}{2} j}^{l}, U_{i+\frac{1}{2} j}^{r}\right)-F\left(U_{i-\frac{1}{2} j}^{l}, U_{i-\frac{1}{2} j}^{r}\right)\right]  \tag{23}\\
& -\frac{\Delta t}{\Delta y}\left[G\left(U_{i j+\frac{1}{2}}^{b}, U_{i j+\frac{1}{2}}^{t}\right)-G\left(U_{i j-\frac{1}{2}}^{b}, U_{i j-\frac{1}{2}}^{t}\right)\right]+\Delta t S_{i j}^{n+\frac{1}{2}}
\end{align*}
$$

Here $t$ and $b$ represent the states at the top and bottom of the cell for the $G$ flux, analogous to the right and left states, resp.

Although the Davis method is not as accurate as other solvers, its ease of use for systems with sources and for systems in several spatial dimensions, and its small operation count, recommend its use for our equations. We note that this approach avoids a splitting of the source terms from the propagation terms. Splitting often creates difficulties for quasisteady flows. In particular, unless special precautions are taken, many hyperbolic solvers (including the Davis solver) introduces dissipation that can destroy special time-independent solutions of the steady-state equations.

We owe the reader a definition of the flux limiting term. We illustrate limiting in the $x$-direction (and drop the $y$-subscript here). Define the forward, backward and centered differences as

$$
\begin{align*}
\Delta_{+} U & =\left(U_{i+1}-U_{i}\right) / \Delta x  \tag{24}\\
\Delta_{-} U & =\left(U_{i}-U_{i-1}\right) / \Delta x  \tag{25}\\
\Delta_{0} U & =\frac{1}{2}\left(U_{i+1}-U_{i-1}\right) / \Delta x \tag{26}
\end{align*}
$$

Then the limited state is

$$
\Delta_{x} U=\frac{1}{2}\left(\operatorname{sgn}\left(\Delta_{+} U\right)+\operatorname{sgn}\left(\Delta_{-} U\right)\right) \min \left(\left|\Delta_{+} U\right|,\left|\Delta_{-} U\right|, 2\left|\Delta_{0} U\right|\right)
$$

Here the sgn ensures that the slope is set to zero at an extrema. The minimum modulus prevents overshoots and undershoots that plague convential higher order methods for nonlinear hyperbolic solvers.

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