POLYNOMIAL CHAOS QUADRATURE-BASED MINIMUM VARIANCE APPROACH FOR SOURCE PARAMETERS ESTIMATION

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International Conference on Computational Science
June 4–June 6, 2012
Sponsors: AFOSR FA9550-11-0336, NSF-CMMI- 1054759, NSF-CMMI-1131074

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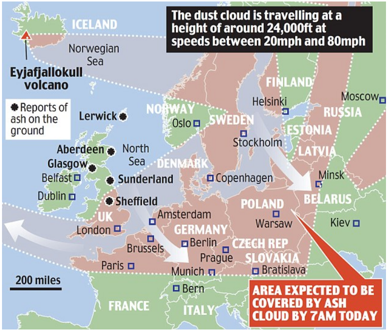
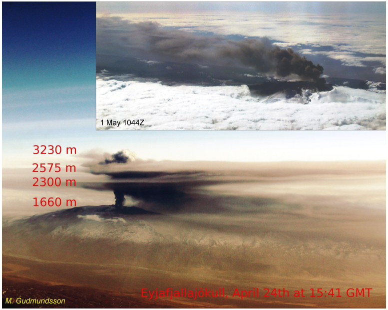


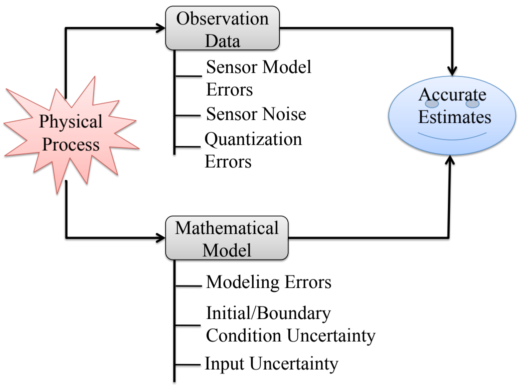
(a)



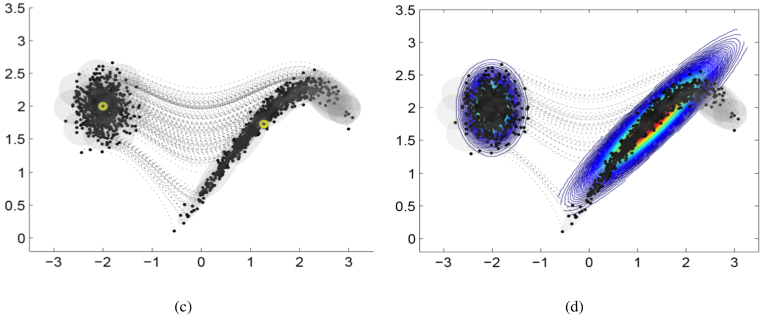
(b)

- The eruption at Eyjafjallajökull, Iceland, had wreaked havoc on European aviation since ash emissions began on 14 April 2010.
- London VAAC began issuing code Red VAAs on 16 April 2010, which was rapidly followed by complete shutdown of British airspace.
- Over the next hours and days, airspace over all of Europe was shut down. Losses were estimated at over $200 million/day (over $1 billion in total), with almost 7 million stranded passengers.

- The BENT integral eruption column model was used to produce eruption column parameters (mass loading, column height, grain size distribution) given *a specific atmospheric sounding and source conditions.*
    - BENT takes into consideration atmospheric (wind) conditions as given by atmospheric sounding data.
    - Plume rise height is given as *a function of volcanic source and environmental conditions.*
- The PUFF Lagrangian model was used to propagate ash parcels in *a given wind field (NCEP Reanalysis).*
    - PUFF takes into account dry deposition as well as dispersion and advection.
- BENT−PUFF Model: $\dot{\mathbf{x}} = f(\Theta, \mathbf{x}, \mathbf{w})$.
    - Uncertain volcano source parameters: vent radius, vent velocity, mean particle size and particle size variance.



- The fusion of observational data with state models promises to provide greater understanding of physical phenomenon than either approach alone can achieve.
- The most critical challenge here is to provide a quantitative assessment of how closely our estimates reflect reality in the presence of model uncertainty as well as measurement errors and uncertainty.



- Approximate Solution to exact problem: Kolmogorov Equation, Monte Carlo Methods, generalized Polynomial Chaos (gPC).
- Exact solution to approximate problem: Gaussian Closure, Equivalent Linearization, and Stochastic Averaging.

POLYNOMIAL CHAOS

- Polynomial chaos is a term originated by *Norbert Wiener in* 1938, to describe the members of the span of Hermite polynomial functionals of a Gaussian process.
- This has been generalized by Xiu et al. (2002) to efficiently use appropraite orthogonal polynomials to model various probability distributions.
  - For example, Hermite polynomials for Gaussian r.v. and Legendre polynomials for uniform r.v.
- For non-polynomial nonlinearity such as transcendental or exponential functions, difficulties may arise during the computation of projection integrals.
- Dalbey *et al.* (2008) have proposed a formulation known as Polynomial Chaos Quadrature (PCQ).

Polynomial Chaos

- PCQ replaces the projection step of the PC with numerical quadrature.

$$\mathscr{E}[x_i(t)^N] = \int_\Omega \left(\int_{t_0}^{t} \dot{x}_i dt \right)^N dp(\boldsymbol{\xi}) = \int_\Omega \left(x_i(t_0, \boldsymbol{\xi}) + \int_{t_0}^{t} \mathbf{f}_i(t, \mathbf{x}, \boldsymbol{\Theta}) dt \right)^N dp(\boldsymbol{\xi})$$
$$= \sum_q w_q \left[\mathscr{X}_i(t_0, t, \boldsymbol{\xi}_q) \right]^N \quad i = 1, 2, \cdots, n \qquad (1)$$

The Polynomial Chaos (PC) expansion for the state vector **x** can be written as:

$$x_i(t, \boldsymbol{\Theta}) = \sum_{k=0}^{N} x_{i_k}(t) \phi_k(\boldsymbol{\xi}) = \mathbf{x}_i^T(t) \boldsymbol{\Phi}(\boldsymbol{\xi}) \Rightarrow \mathbf{x}(t, \boldsymbol{\xi}) = \mathbf{X}_{pc}(t) \Phi(\boldsymbol{\xi})$$
$$x_{i_k}(t) = \frac{1}{d_k^2} \sum_{q=1}^{M} \mathscr{X}_i(t_0, t, \boldsymbol{\xi}_q) \phi_k(\boldsymbol{\xi}_q) w_q, \quad k = 0, \cdots, N, \;\; i = 1, \cdots, n \qquad (2)$$

BAYES' RULE

$$p(\boldsymbol{\Theta},\mathbf{x}|\mathbf{Y}_k) = \frac{p(\boldsymbol{\Theta},\mathbf{x}|\mathbf{Y}_{k-1})p(\mathbf{y}(t_k)|\boldsymbol{\Theta},\mathbf{x})}{p(\mathbf{y}(t_k))} \quad (3)$$

POSTERIOR CONDITIONAL MOMENTS OF A SCALAR FUNCTION $\phi(\boldsymbol{\Theta},\mathbf{x})$

$$\hat{\phi}_k^+ = \frac{\mathscr{E}^-[\phi(\boldsymbol{\Theta},\mathbf{x})p(\mathbf{y}_k|\boldsymbol{\Theta},\mathbf{x})]}{p(\mathbf{y}_k)} \quad (4)$$

$$\hat{\phi}_k^+ = \mathscr{E}^+[\phi(\boldsymbol{\Theta},\mathbf{x})] \triangleq \int\int \phi(\boldsymbol{\Theta},\mathbf{x})p(\boldsymbol{\Theta},\mathbf{x}|\mathbf{Y}_k)d\boldsymbol{\Theta}d\mathbf{x} \quad (5)$$

$$\hat{\phi}_k^- = \mathscr{E}^-[\phi(\boldsymbol{\Theta},\mathbf{x})] \triangleq \int\int \phi(\boldsymbol{\Theta},\mathbf{x})p(\boldsymbol{\Theta},\mathbf{x}|\mathbf{Y}_{k-1})d\boldsymbol{\Theta}d\mathbf{x} \quad (6)$$

- Representing prior PC coefficients by $\boldsymbol{\Theta}_{pc}^-$ and $\mathbf{X}_{pc}^-$ and posterior PC coefficients by $\boldsymbol{\Theta}_{pc}^+$ and $\mathbf{X}_{pc}^+$, $\hat{\phi}^-$ and $\hat{\phi}^+$ can be written as:

$$\hat{\phi}_k^- = \mathscr{E}^-[\phi(\boldsymbol{\Theta}, \mathbf{x})] = \int \phi(\boldsymbol{\Theta}_{pc}^- \Phi(\boldsymbol{\xi}), \mathbf{X}_{pc}^-(t)\Phi(\boldsymbol{\xi}))p(\boldsymbol{\xi})d\boldsymbol{\xi} \qquad (7)$$

$$\hat{\phi}_k^+ = \mathscr{E}^+[\phi(\boldsymbol{\Theta}, \mathbf{x})] = \int \phi(\boldsymbol{\Theta}_{pc}^+ \Phi(\boldsymbol{\xi}), \mathbf{X}_{pc}^+(t)\Phi(\boldsymbol{\xi}))p(\boldsymbol{\xi})d\boldsymbol{\xi} \qquad (8)$$

PRIOR AND POSTERIOR MEAN

$$\hat{\mathbf{z}}_k^- \triangleq \mathscr{E}^-[\mathbf{z}_k] = \begin{bmatrix} \mathbf{X}_{pc_1}^-(t) \\ \boldsymbol{\Theta}_{pc_1}^- \end{bmatrix}, \ \mathbf{z}(t, \boldsymbol{\xi}) = \begin{bmatrix} \mathbf{x}(t, \boldsymbol{\xi}) \\ \boldsymbol{\Theta}(\boldsymbol{\xi}) \end{bmatrix} \qquad (9)$$

$$\hat{\mathbf{z}}_k^+ \triangleq \mathscr{E}^+[\mathbf{z}_k] = \begin{bmatrix} \mathbf{X}_{pc_1}^+(t) \\ \boldsymbol{\Theta}_{pc_1}^+ \end{bmatrix} \qquad (10)$$

Prior and Posterior Covariance

$$\boldsymbol{\Sigma}_k^- \triangleq \mathscr{E}^-[(\mathbf{z}_k - \hat{\mathbf{z}}_k^-)(\mathbf{z}_k - \hat{\mathbf{z}}_k^-)^T] = \begin{pmatrix} \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{-2} & \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{-} \boldsymbol{\Theta}_{pc_i}^{-} \\ \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{-} \boldsymbol{\Theta}_{pc_i}^{-} & \sum_{i=1}^{N} \boldsymbol{\Theta}_{pc_i}^{-2} \end{pmatrix} \tag{11}$$

$$\boldsymbol{\Sigma}_k^+ \triangleq \mathscr{E}^+[(\mathbf{z}_k - \hat{\mathbf{z}}_k^-)(\mathbf{z}_k - \hat{\mathbf{z}}_k^-)^T] = \begin{pmatrix} \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{+2} & \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{+} \boldsymbol{\Theta}_{pc_i}^{+} \\ \sum_{i=1}^{N} \mathbf{X}_{pc_i}^{+} \boldsymbol{\Theta}_{pc_i}^{+} & \sum_{i=1}^{N} \boldsymbol{\Theta}_{pc_i}^{+2} \end{pmatrix} \tag{12}$$

Minimum Variance Estimator

$$\hat{\mathbf{z}}_k^+ = \hat{\mathbf{z}}_k^- + \mathbf{K}_k[\tilde{\mathbf{y}}_k - \mathscr{E}^-[\mathbf{h}(\mathbf{x}_k)]] \tag{13}$$

$$\mathbf{\Sigma}_k^+ = \mathbf{\Sigma}_k^- + \mathbf{K}_k\mathbf{\Sigma}_{zy} \tag{14}$$

$$\mathbf{K}_k = -\mathbf{\Sigma}_{zy}^T\left(\mathbf{P}_{hh}^- + \mathbf{R}_k\right)^{-1} \tag{15}$$

$$\hat{\mathbf{h}}_k^- \triangleq \mathscr{E}^-[\mathbf{h}(\mathbf{x}_k, \boldsymbol{\Theta})] = \sum_{q=1}^{M} w_q \underbrace{\mathbf{h}(\mathbf{x}_k(\boldsymbol{\xi}_q))}_{\mathbf{h}_q}$$

$$\mathbf{\Sigma}_{zy} \triangleq \mathscr{E}^-[(\mathbf{z}_k - \hat{\mathbf{z}}_k)(\mathbf{h}(\mathbf{x}_k) - \hat{\mathbf{h}}_k^-)^T] = \sum_{q=1}^{M} w_q(\mathbf{z}_k(\boldsymbol{\xi}_q) - \hat{\mathbf{z}}_k^-)(\mathbf{h}_q - \hat{\mathbf{h}}_k^-)^T$$

$$\mathbf{\Sigma}_{hh}^- \triangleq \mathscr{E}^-[(\mathbf{h}(\mathbf{x}_k) - \hat{\mathbf{h}}_k^-)(\mathbf{h}(\mathbf{x}_k) - \hat{\mathbf{h}}_k^-)^T] = \sum_{q=1}^{M} w_q(\mathbf{h}_q - \hat{\mathbf{h}}_k^-)(\mathbf{h}_q - \hat{\mathbf{h}}_k^-)^T$$

R. Madankan, P. Singla, T. Singh and P. Scott, "Polynomial Chaos Based Method for State and Parameter Estimation,"

2012 American Control Conference, Montréal, Canada, June 27-June 29, 2012.

TABLE: Eruption source parameters based on observations of Eyjafjallajökull volcano and information from other similar eruptions.

Parameter	Value range	PDF	Comment
Vent radius, b_0, m	65-150	Uniform	Measured from radar image of summit vents
Vent velocity, w_0, m/s	Range: 45-124	Uniform	M. Ripepe, Geneva, Switzerland, 2010, presentation
Mean grain size, Md_φ	2 boxcars: 1.5-2 and 3-5	Multi-Modal Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian. A. Hoskuldsson, Iceland meeting 2010, presentation
σ_φ	1.9 ± 0.6	Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian.

Prior Mean Ash Top Height

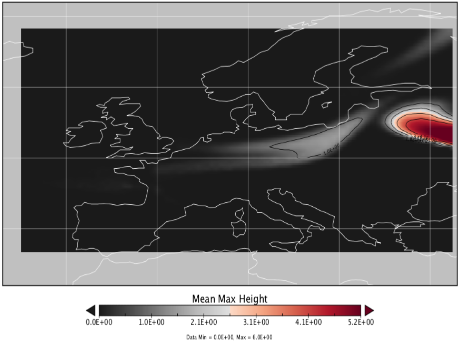


Figure: 9^4 *Clenshaw Curtis Runs*

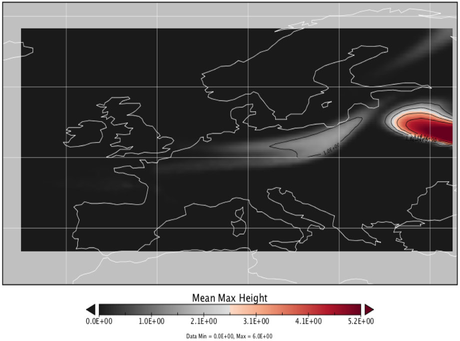
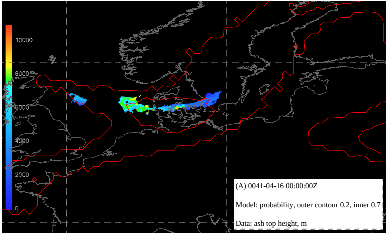
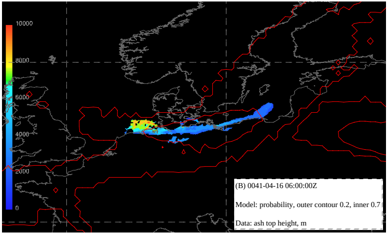


Figure: 161 *CUT Runs*

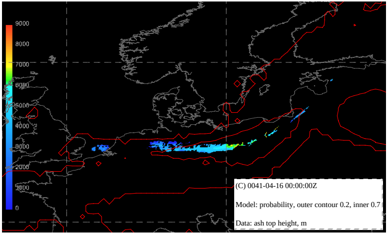
N. Adhurthi, P. Singla and T. Singh, "The Conjugate Unscented Transform - An Approach to Evaluate Multi- Dimensional Expectation Integrals," *2012 American Control Conference, Montréal, Canada, June 27-June 29, 2012.*



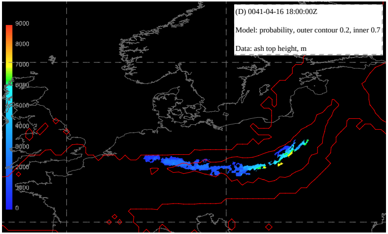
(a) 00 hrs



(b) 06 hrs

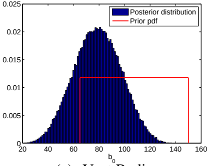


(c) 12 hrs

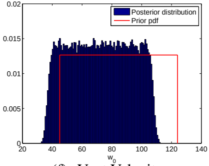


(d) 18 hrs

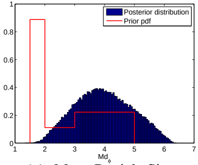
- Posterior distribution of source parameters, using Actual satellite observed ash top-height data:



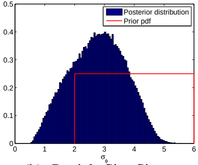
(e) Vent Radius



(f) Vent Velocity

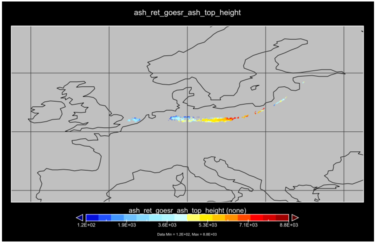


(g) Mean Particle Size

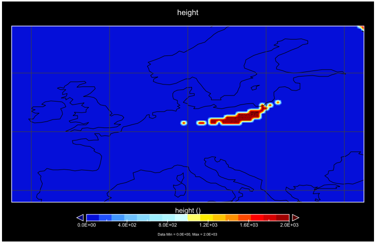


(h) Particle Size Sigma

Figure: *Posterior distribution of Source Parameters, using actual satellite data*



(a) April 16^{th}, 1200 hrs, CALIPSO Image



(b) April 16^{th}, 1200 hrs, Bent-Puff

- PCQ based minimum variance estimator is developed for volcano source estimation.
  - Four source parameters are considered as uncertain variables.
  - We were able to predict ash footprint with good confidence even though source uncertainty is large.
  - Can be applied as a batch or recursive estimation techniques.
  - Posterior distribution of source parameters can be obtained.
- The PCQ approach uses `BENT` and `PUFF` as black-box models
  - Any other models for ash dispersion can be included in the uncertainty analysis.
- Work in Progress:
  - Effect of uncertainty in wind data still needs to be analyzed.
  - Resolution of model vs. data resolution.
  - Likelihood functions: validating the sensor data.
  - *uniqueness of the parameters.*

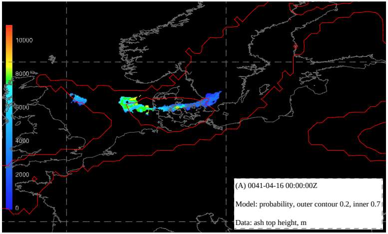


Figure: *Pr(Top Ash Height)* $\geq$ *0km*

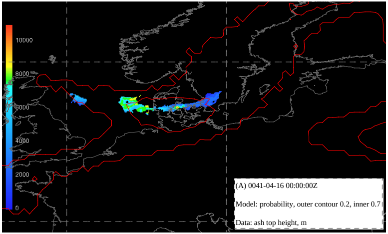


Figure: *Pr(Top Ash Height)* $\geq$ *2km*

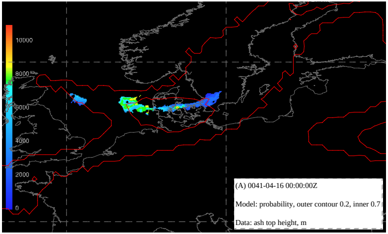


Figure: *Pr(Top Ash Height)* $\geq$ *4km*

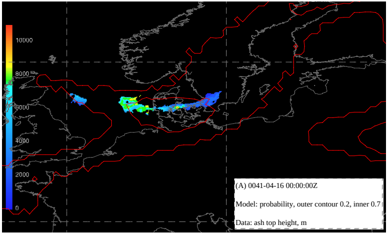
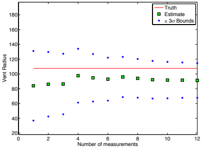
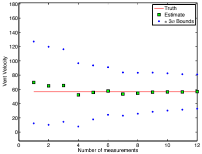


Figure: *Pr(Top Ash Height)* $\geq$ *6km*

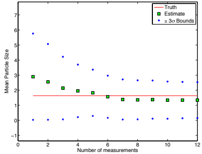
- Estimation of source parameters, using synthetic satellite observed ash top-height data:



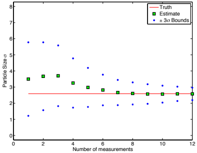
(a) Vent Radius



(b) Vent Velocity



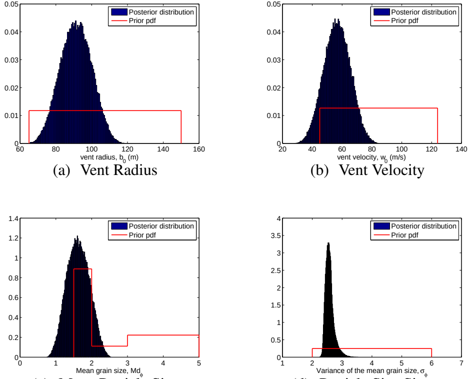
(c) Mean Particle Size



(d) Particle Size Sigma

Figure: *Estimated Source Parameters, using synthetic satellite data*

- Posterior distribution of source parameters, using synthetic satellite observed ash top-height data:



(a) Vent Radius (b) Vent Velocity

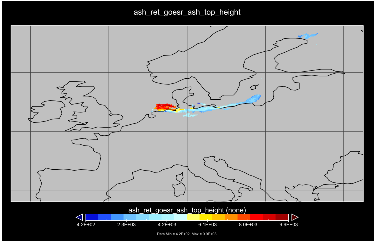
(c) Mean Particle Size (d) Particle Size Sigma

Figure: *Posterior distribution of Source Parameters, using synthetic satellite data*

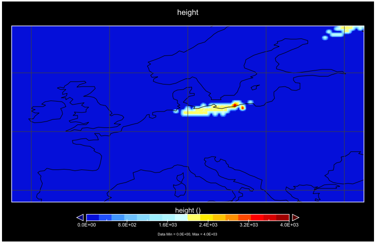
- Estimation of source parameters, using actual satellite observed ash top-height data:

Table: Posterior mean (z_p) and covariance (P_p) of source parameters obtained from actual satellite top-height data

	Using Apr.16th, 0000, 0600 & 1200 hrs
$z_p = [b_0, w_0, Md_\phi, \sigma_\phi]$	$[84.848, 63.559, 1.711, 1.736]$
$P_p = E[z_p^T z_p]$	$\begin{pmatrix} 323.08 & -142.49 & -0.38 & -1.35 \\ -142.49 & 394.89 & -0.30 & -0.93 \\ -0.39 & -0.30 & 0.87 & 0.19 \\ -1.35 & -0.93 & 0.19 & 0.77 \end{pmatrix}$

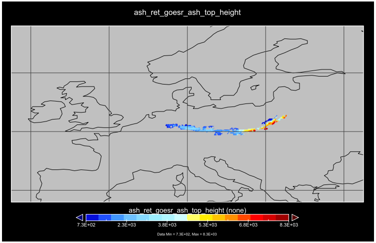


(a) Satellite

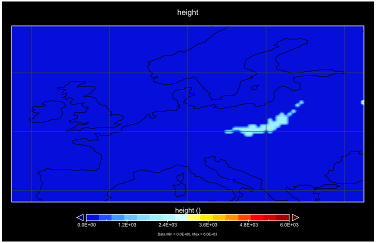


(b) Bent-Puff

Figure: *April* 16^{th}, *0600 hrs*

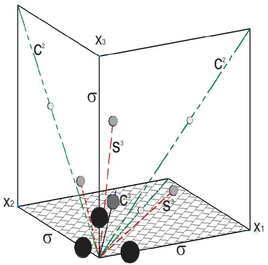


(a) Satellite

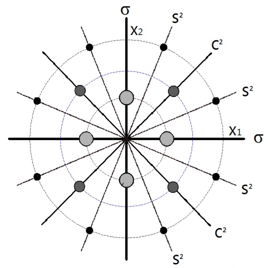


(b) Bent-Puff

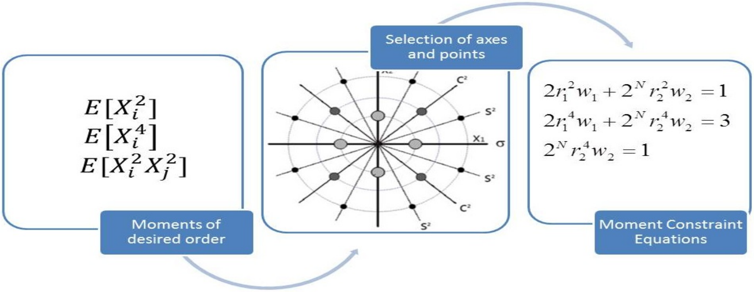
Figure: *April* 16^{th}, *1800 hrs*



X3
C²
C²
σ
S³
X2
C³
S³
X1
σ
σ



σ
X2
S²
C²
S²
X1
σ
S²
C²
S²



Selection of axes and points
$E[X_i^2]$
$E[X_i^4]$
$E[X_i^2 X_j^2]$
Moments of desired order
$2r_1^2 w_1 + 2^N r_2^2 w_2 = 1$
$2r_1^4 w_1 + 2^N r_2^4 w_2 = 3$
$2^N r_2^4 w_2 = 1$
Moment Constraint Equations

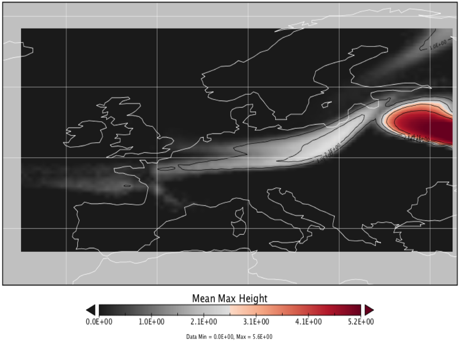


Figure: 9^4 *Clenshaw Curtis Runs*

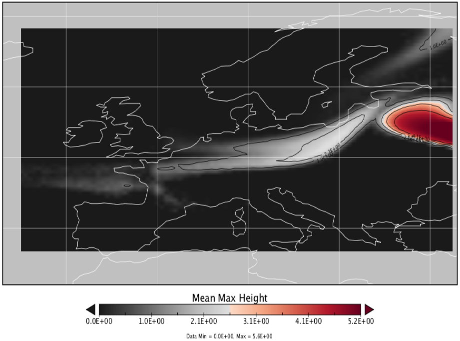


Figure: 161 *CUT Runs*