

VOLCANIC DEFORMATION MODELLING: NUMERICAL BENCHMARKING WITH COMSOL

The following is a description of the model setups and input/output parameters for benchmarking analytical volcanic deformation models with the commercial Finite Element Analysis package COMSOL Multiphysics, version 4.2a. They should be easily convertible for earlier, or indeed later, versions.

1. The ‘Mogi’ Model

In the ‘Mogi’ model, the deformation source is represented by a small point embedded in a homogeneous, isotropic, elastic half-space [Mogi, 1958], and the resultant surface displacements are defined by:

$$\begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = \alpha^3 \Delta P \frac{(1 - \nu)}{G} \begin{pmatrix} \frac{x}{R^3} \\ \frac{y}{R^3} \\ \frac{d}{R^3} \end{pmatrix} \quad (\text{Eq. 1})$$

where U_x , U_y and U_z are the displacements at the point $(x, y, 0)$ on the free surface, the point source is at $(0, 0, -d)$, α and ΔP are the radius and pressure change of the source respectively, ν is the Poisson ratio, G is the shear modulus (or rigidity) and $R = \sqrt{(x^2 + y^2 + d^2)}$; the distance from the centre of the source to a point on the free surface.

1.1. 2D Axi-symmetric

- ❖ Open a 2D axi-symmetric model in the structural mechanics module using the basic stationary solver.
- ❖ Build the geometry for the model domain to replicate a half-space:
 - A square with its top boundary at $z = 0$ (the free surface), its left boundary at $r = 0$ (the axis of symmetry), and of sufficient size such that the boundary conditions do not effect the results of the interior.
 - A circle with its centre at $r = 0, z = -d$, and its radius defined. N.B. The radius must be much smaller than the depth (d).
 - Select both the square and the circle and press the ‘Difference’ tab so that you are left with a square containing a semi-circle cavity.
- ❖ Define the material (default is homogeneous, elastic and isotropic):

- Right-click to add a material to the entire model domain and specify the choice of Young's Modulus and Poisson's Ratio. The choice of density will not affect the surface deformation results.
- ❖ Set the model physics:
 - Add a 'roller' to the right boundary of the square to prevent deformation perpendicular to the boundary.
 - Add a 'prescribed deformation' condition to the bottom boundary to prevent any deformation.
 - The top boundary of the square should be 'free' by default.
 - Add a 'boundary load' to the two boundaries making the semi-circle.
 - Under 'load type' select pressure and define the necessary pressure value.
- ❖ Define the mesh:
 - Add the mesh shape of choice (default is free triangular).
 - Set the mesh size to a detail sufficient to resolve necessary displacements. N.B. Pay particular attention to the areas around the source boundaries and the free surface.
- ❖ Run the model.
- ❖ Select the desired data for display:
 - For a line plot of deformation at the free surface add a 1D plot group and then add a line graph. Select the free surface from the boundary selection box and then choose the vertical and/or radial displacement to display.
- ❖ Export data for further analysis:
 - For the deformation data along the free surface add a '2D cut line' under 'Data Sets'. Then under export, add 'Data' and choose the 2D cut line, then add the data sets you wish to use, e.g. vertical and/or radial deformation.

N.B. In COMSOL v3.5 the boundary load must be applied with the tangent and normal coordinate system.

1.2. 3D

- ❖ Open a 3D model in the structural mechanics module using the basic stationary solver.
- ❖ Build the geometry for the model domain to replicate a half-space:
 - A block with its top boundary at $z = 0$ (the free surface) and of sufficient size such that the boundary conditions do not effect the results of the interior.

- A sphere in the middle of the block, with its centre at $x = \text{half the } x \text{ dimension of the block}$, $y = \text{half the } y \text{ dimension of the block}$, $z = -d$, and its radius defined. N.B. The radius must be much smaller than the depth (d).
 - Select both the block and the sphere and press the ‘Difference’ tab so that you are left with a block containing a spherical cavity.
- ❖ Define the material (default is homogeneous, elastic and isotropic):
 - Right-click to add a material to the entire model domain and specify the choice of Young’s Modulus and Poisson’s Ratio. The choice of density will not affect the surface deformation results.
- ❖ Set the model physics:
 - Add ‘rollers’ to the lateral boundary surfaces of the block to prevent deformation perpendicular to the boundary.
 - Add a ‘prescribed deformation’ condition to the bottom boundary surface to prevent any deformation.
 - The top boundary surface of the block should be ‘free’ by default.
 - Add a ‘boundary load’ to the eight boundary surfaces making the sphere.
 - Under ‘load type’ select pressure and define the necessary pressure value.
- ❖ Define the mesh:
 - Add the mesh shape of choice (default is free tetrahedral).
 - Set the mesh size to a detail sufficient to resolve necessary displacements. N.B. Pay especially particular attention to the areas around the source boundaries and the free surface.
- ❖ Run the model.
- ❖ Select the desired data for display:
 - For a line plot of deformation at the free surface add a 1D plot group and then add a line graph. Under ‘Data Sets’ add a ‘3D cut line’ along the necessary transect. Select the ‘3D cut line’ as the data set in the line graph tab and then choose the vertical and/or radial displacement to display.
- ❖ Export data for further analysis:
 - Under export, add ‘Data’ and choose the 3D cut line created for the plot. Then add the data sets you wish to use, e.g. vertical and/or radial deformation.

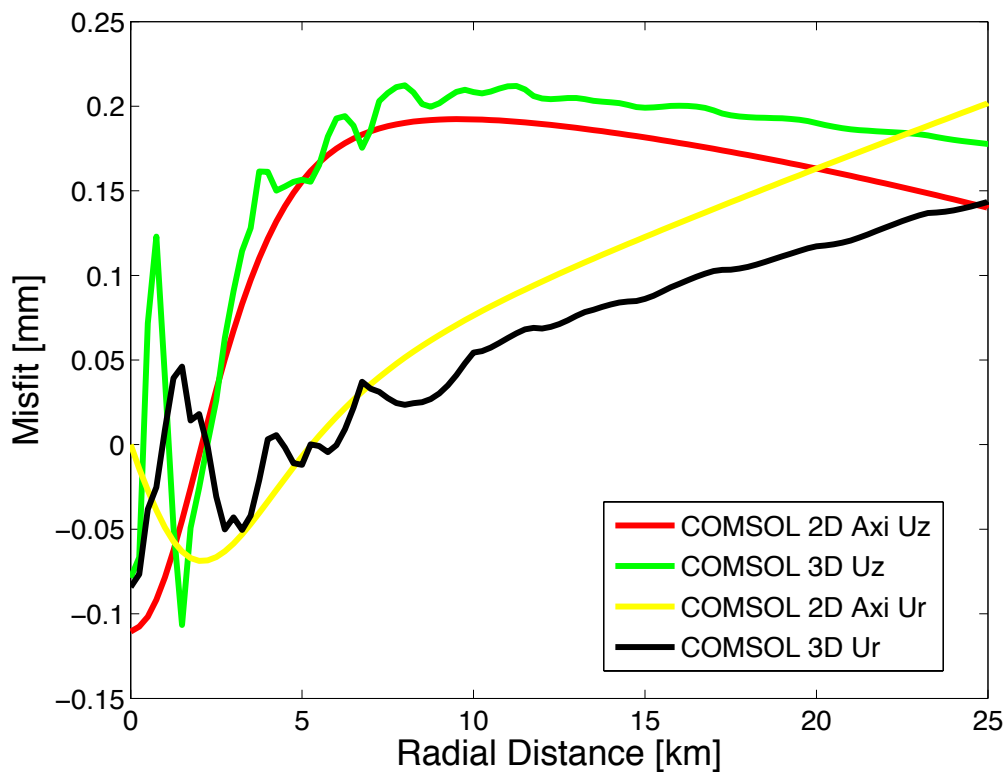
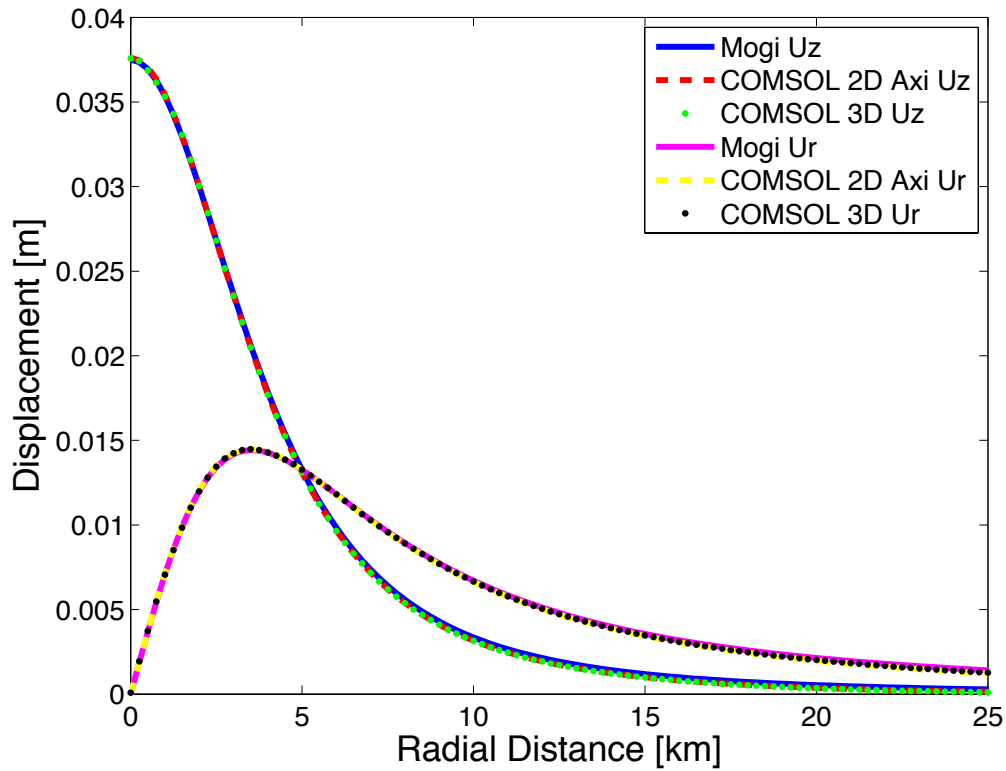


Figure 1: Calibration of the benchmark FEM's with the analytical solution of a point source [Mogi, 1958]. (Top) The three are nearly perfectly calibrated in both vertical (Uz) and horizontal displacement (Ur). (Bottom) Slight mismatches between the numerical and analytical solutions are thought to be artefacts of the mesh density.

2. Spherical source in a viscoelastic half-space – *Del Negro et al., 2009*

To solve for the displacement induced by a spherical source in a viscoelastic half-space, *Del Negro et al.*, [2009] applied the Laplace transform and the Correspondence Principle to the analytical solutions of *Mogi* [1958]. Their model assumes a generalized Maxwell rheology with one Maxwell viscoelastic branch in parallel with a purely elastic branch. The resultant surface displacements at time, t , are calculated by:

$$\begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = \alpha^3 \Delta P \begin{pmatrix} \frac{x}{R^3} \\ \frac{y}{R^3} \\ \frac{d}{R^3} \end{pmatrix} \times A(t) \quad (\text{Eq. 2})$$

and

$$A(t) = \frac{1}{2G_0} \left[\frac{3K + 4G_0\mu_0}{\mu_0(3K + G_0\mu_0)} - 3 \frac{\eta G_0^2 e^{-\left(\frac{G_0\mu_1(3K+G_0\mu_0)}{\eta(3K+G_0)}\right)t}}{\eta(3K + G_0\mu_0)(3K + G_0)} \times (1 - \mu_0) - \left(\frac{1}{\mu_0} - 1\right) e^{-\left(\frac{G_0\mu_0\mu_1}{\eta}\right)t} \right] \quad (\text{Eq. 3})$$

where U_x , U_y and U_z are the displacements at the point $(x, y, 0)$ on the free surface, the source centre is at $(0, 0, -d)$, α and ΔP are the radius and pressure change of the source respectively, and $R = \sqrt{(x^2 + y^2 + d^2)}$; the distance from the centre of the source to a point on the free surface. K is the Bulk Modulus and is constant since the viscous part of the deformation is incompressible, and η is the viscosity. G_0 refers to the total shear modulus, which is split equally between the two branches via the fractional shear moduli, μ ($\mu_0 = \mu_1 = 0.5$). The viscoelastic response then depends on the Maxwell relaxation time, $\tau_0 = (G_0\mu_1)$.

2.1. 2D Axi-symmetric

- ❖ Open a 2D axi-symmetric model in the structural mechanics module using the basic time-dependent solver.
- ❖ Build the geometry for the model domain to replicate a half-space:
 - A square with its top boundary at $z = 0$ (the free surface), its left boundary at $r = 0$ (the axis of symmetry), and of sufficient size such that the boundary conditions do not effect the results of the interior.

- A circle with its centre at $r = 0$, $z = -d$, and its radius defined.
 - Select both the square and the circle and press the ‘Difference’ tab so that you are left with a square containing a semi-circle cavity.
- ❖ Set the model physics:
- Add a ‘roller’ to the right boundary of the square to prevent deformation perpendicular to the boundary.
 - Add a ‘prescribed deformation’ condition to the bottom boundary to prevent any deformation.
 - The top boundary of the square should be ‘free’ by default.
 - Add a ‘boundary load’ to the two boundaries making the semi-circle.
 - Under ‘load type’ select pressure and define the necessary pressure value.
 - Add a ‘viscoelastic material’ to the entire model domain and click to specify the Bulk Modulus and Shear Modulus as the material properties. These will be defined later.
 - Under the ‘linear elastic’ tab select the ‘force linear strains’ box.
 - In the ‘solid mechanics’ node choose ‘quasi-static’ for the ‘structural transient behaviour’.
- ❖ Define the material:
- Right-click to add a material to the entire model domain and specify the choice of Bulk Modulus.
 - The value for Shear Modulus is split between the two branches of the Maxwell model (one elastic, one viscoelastic), therefore, after calculating the Shear Modulus, enter half of its value here, e.g. if $G = 30$ GPa only enter 15 GPa.
 - The choice of density will not affect the surface deformation results.
- ❖ Set the viscoelastic parameters:
- Under the ‘viscoelastic material model’ node enter the other half of the Shear Modulus in branch 1 of the generalized Maxwell model.
 - For the corresponding relaxation time, enter the value (in seconds) equal to the chosen value of viscosity divided by the Shear Modulus of that branch, e.g. if viscosity = $1e17$ Pa s and the branch Shear Modulus = 15 GPa, then the time value = $6.67e6$ s.
- ❖ Define the mesh:
- Add the mesh shape of choice (default is free triangular).
 - Set the mesh size to a detail sufficient to resolve necessary displacements. N.B. Pay particular attention to the areas around the source boundaries and the free surface.
- ❖ Set the time dependency:
- In the study step, open the time dependency node and under ‘study settings’ define the time window you wish to use, i.e. start, stop, and step-length.

- **NOTE:** for the current settings the model fails if very long time windows are chosen relative to the characteristic time defined in the viscoelastic settings, and the solution at $t=0$ is invalidated (it should otherwise calibrate with a purely elastic solution). Work is underway to correct this.
- ❖ Run the model.
- ❖ Select the desired data for display:
 - For a line plot of deformation at the free surface add a 1D plot group and then add a line graph. Select the free surface from the boundary selection box and then choose the vertical and/or radial displacement to display.
- ❖ Export data for further analysis:
 - For the deformation data along the free surface add a '2D cut line' under 'Data Sets'. Then under export, add 'Data' and choose the 2D cut line, then add the data sets you wish to use, e.g. vertical and/or radial deformation.

2.2. 3D

- ❖ Take the viscoelastic procedure described in §2.1. and apply it to the 3D 'Mogi' model set-up in §1.2.

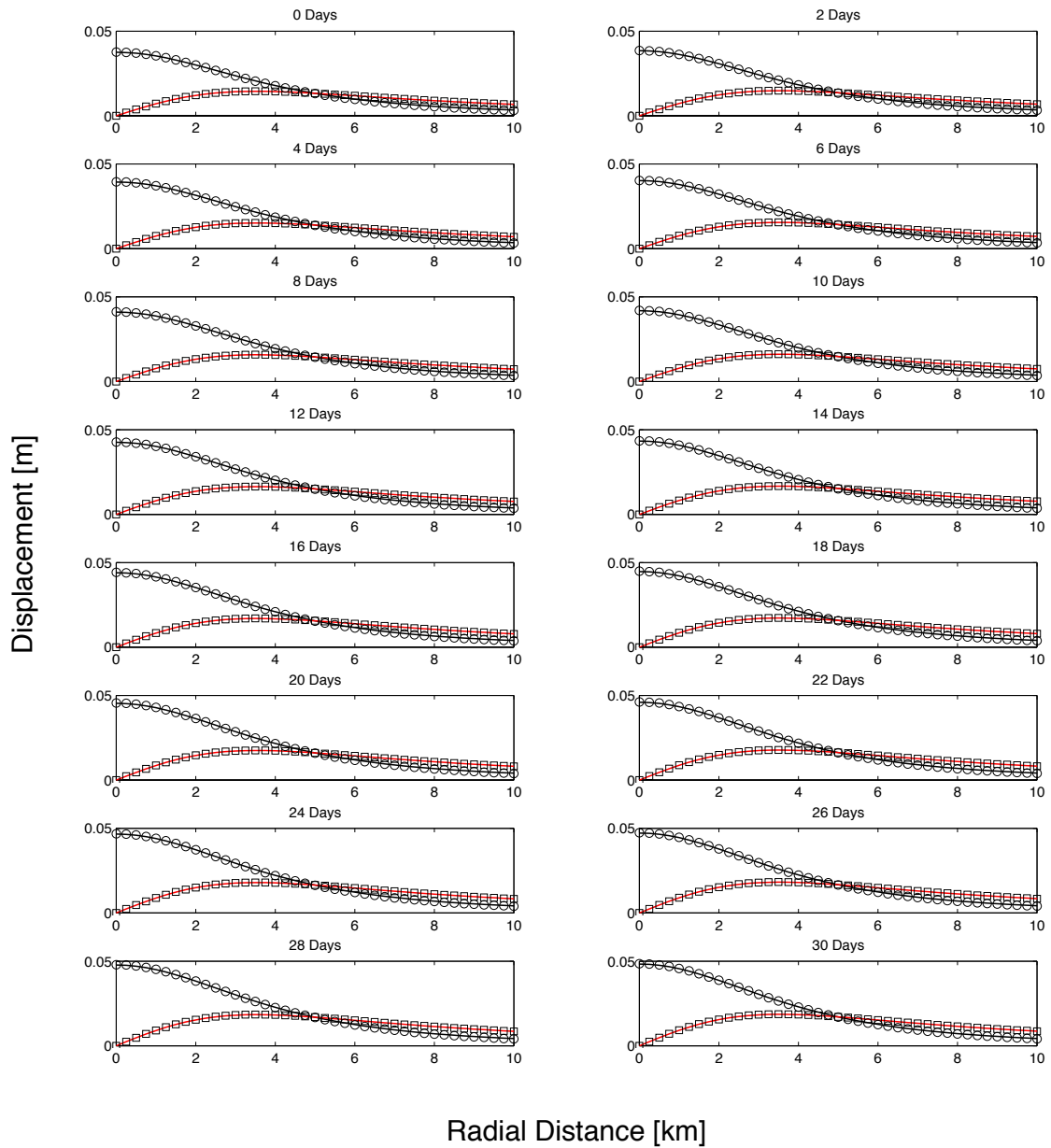


Figure 2: Calibration of the benchmark FEM with the analytical solution of a spherical source in a viscoelastic half-space [Del Negro *et al.*, 2009]. The solid lines are the analytical solutions for vertical displacement (black) and radial displacement (red). The black shapes are the numerical solutions for the vertical displacement (circles) and radial displacement (squares). The two are nearly perfectly calibrated in both vertical and horizontal at all time steps.